Lilac: A Modal Separation Logic for Conditional Probability

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https://johnm.li/lilac.pdf

How to reason about complex probabilistic systems?

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Is my car safe?

How to reason about complex probabilistic systems?

Is my car safe?

Is this decision fair?

How to reason about complex probabilistic systems?

Is my car safe?

Is this decision fair?

Is my result significant?

How to reason about complex probabilistic systems?

- Reasoning should be *modular*:

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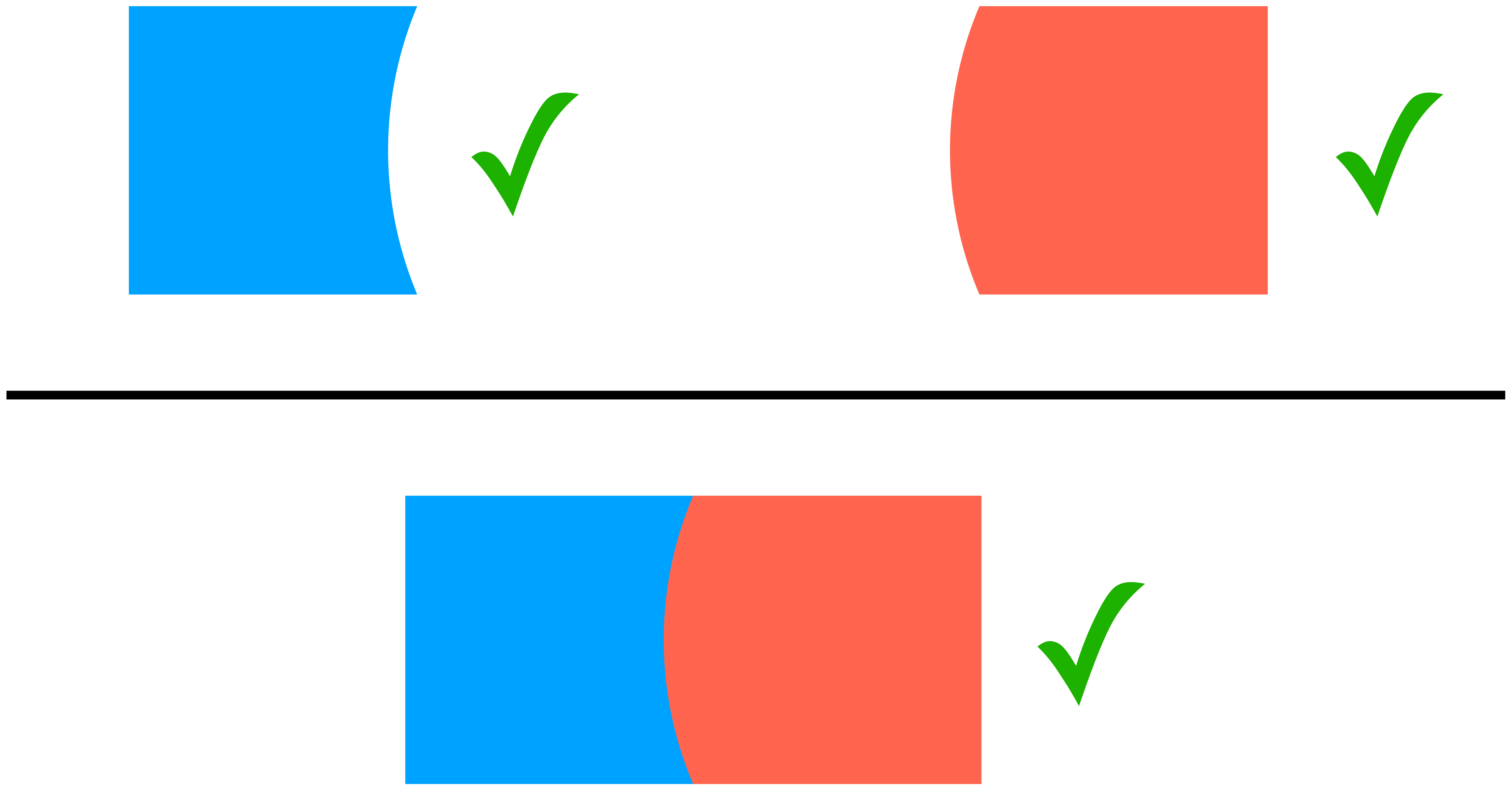
- Reasoning should be *modular*:



Modularity comes from probabilistic independence

- Independence arises frequently and naturally:

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$$\texttt{weights} = \texttt{np.random.rand}(1000)$$

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$$\texttt{weights} = \texttt{np.random.rand}(1000)$$

$$\texttt{weights}[0], \ldots, \texttt{weights}[999] \sim \text{Unif}[0,1] \text{ mutually independent}$$

Modularity comes from probabilistic independence

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$$\texttt{result} = \texttt{np.mean}(\texttt{data})$$

Modularity comes from probabilistic independence

- Independence arises frequently and naturally:

if each `data[i]` is an independent estimate of v...

$$\texttt{result} = \texttt{np.mean(data)}$$

...then `result` is a more accurate estimate of v

Modularity comes from probabilistic independence

- Independence arises frequently and naturally.
- Idea: capture independence using *separation logic*

Ordinary separation logic is about disjointness

$$x = \textsf{new}\ 0\,;$$

$$y = \textsf{new}\ 1\,;$$

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↑

x and y point to disjoint heap locations

Ordinary separation logic is about disjointness

$$\frac{\{P\}\ e\ \{x\,.\,Q(x)\}}{\{P * F\}\ e\ \{x\,.\,Q(x) * F\}}\ \text{(Frame)}$$

Ordinary separation logic is about disjointness

When verifying e...

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When verifying e...

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- This has enabled modular heap-based reasoning at scale.[1]

[1] C. Calcagno and D. Distefano. Infer: An automatic program verifier for memory safety of C programs. NFM 2011.

Lilac's separation is about independence

$$X \leftarrow \texttt{flip}\ 1/2;$$

$$Y \leftarrow \texttt{flip}\ 1/2;$$

Lilac's separation is about independence

$$X \leftarrow \texttt{flip}\ 1/2;$$

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$$X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2)$$

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$$X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2)$$

↑

X and Y are independent random variables

New in Lilac

New in Lilac: a simple frame rule

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- Just like in ordinary separation logic!

New in Lilac: separation is independence

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Inexpressible in prior work

New in Lilac: separation is independence

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$$(\texttt{weights}[0] \sim \text{Unif}[0,1]) \;\ast\; \cdots \;\ast\; (\texttt{weights}[999] \sim \text{Unif}[0,1])$$

↑

Completely captures independence (Lemma 2.5)

New in Lilac: quantitative reasoning

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$$\texttt{result} = \texttt{np.mean}(\texttt{data})$$

...then `result` is a more accurate estimate of v

New in Lilac: quantitative reasoning

if each `data[i]` independent
and for all i we have $\mathbb{E}[\texttt{data[i]}] = v$ and $\mathrm{Var}(\texttt{data[i]}) \leq \varepsilon$...

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New in Lilac: good interop with normal math

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$$\ldots\text{then } \mathbb{E}[\texttt{result}] = v \text{ and } \mathrm{Var}(\texttt{result}) \le \frac{\varepsilon}{|\texttt{data}|}$$

An ordinary random variable

New in Lilac: good interop with normal math

if $\mathop{\Huge *}_{0 \le i < |\mathtt{data}|} \left(\mathbb{E}[\mathtt{data[i]}] = v \text{ and } \mathrm{Var}(\mathtt{data[i]}) \le \varepsilon \right)$...

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Ordinary expectation and variance

New in Lilac: good interop with normal math

if $\mathop{\Huge\ast}_{0 \le i < |\texttt{data}|} \left(\mathbb{E}[\texttt{data[i]}] = v \text{ and } \mathrm{Var}(\texttt{data[i]}) \le \varepsilon \right)$...

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...then $\mathbb{E}[\texttt{result}] = v$ and $\mathrm{Var}(\texttt{result}) \le \dfrac{\varepsilon}{|\texttt{data}|}$

$\implies$ textbook proofs remain textbook

Key idea

Key idea

- Probability spaces are the heaps of probability theory.

Probability spaces as heaps

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$$X \sim \mathrm{Ber}(1/2)$$

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$X \sim \mathrm{Ber}(1/2)$ means $\Pr[X = \mathrm{true}] = \Pr[X = \mathrm{false}] = 1/2$

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This hides a lot of machinery...

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$X \sim \mathrm{Ber}(1/2)$ really means...

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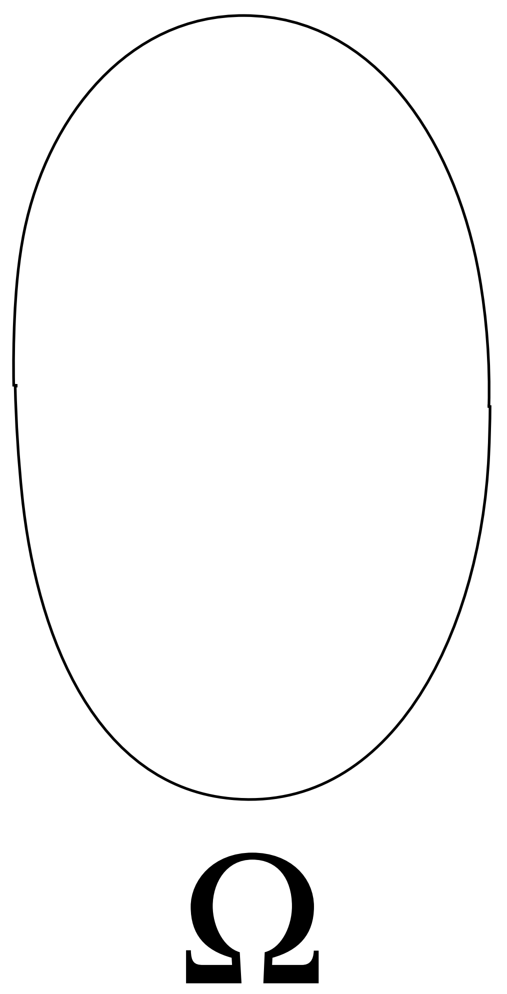
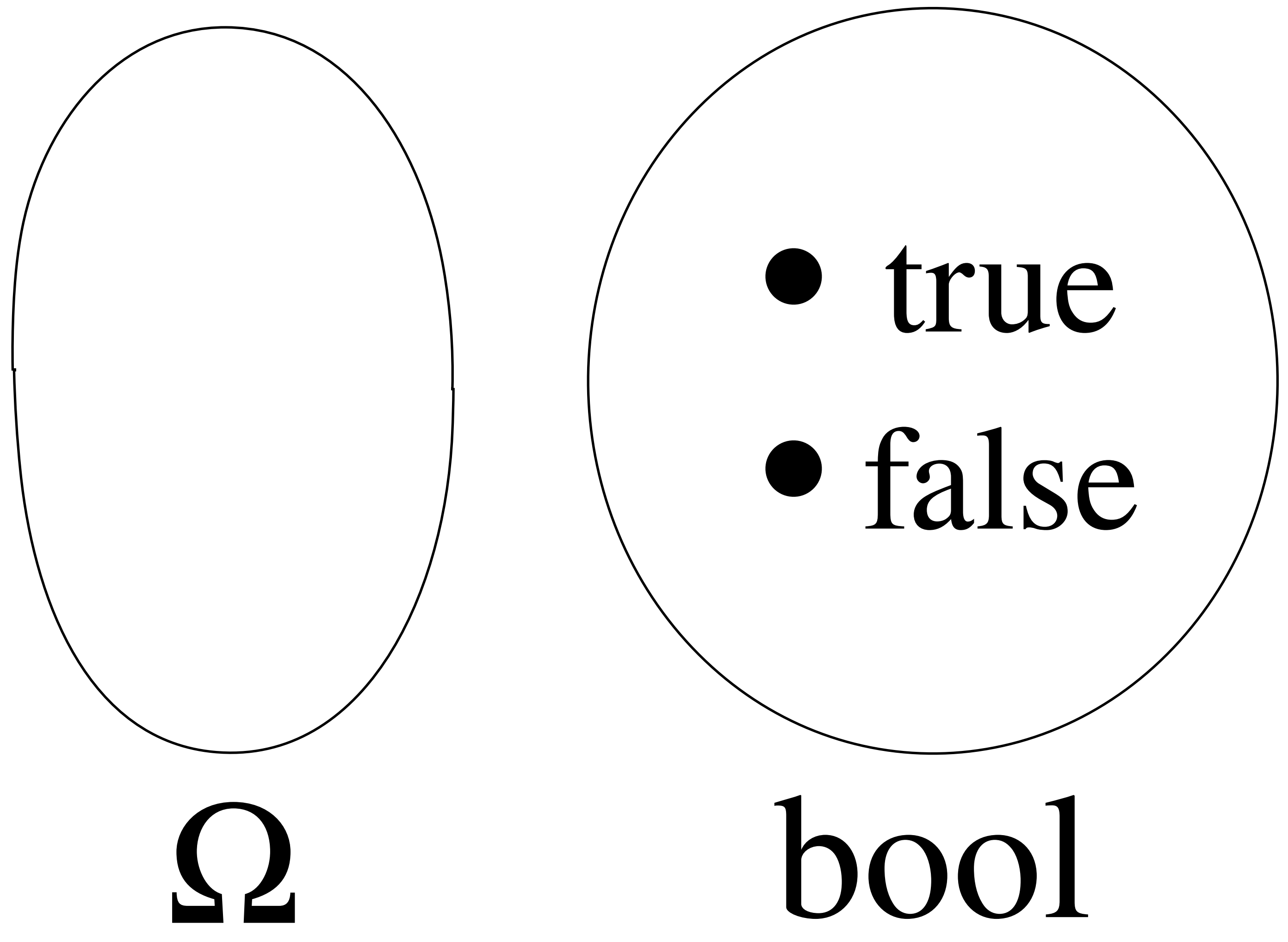


Probability spaces as heaps

$X \sim \mathrm{Ber}(1/2)$ really means...

$$X : \Omega \to \mathrm{bool}$$



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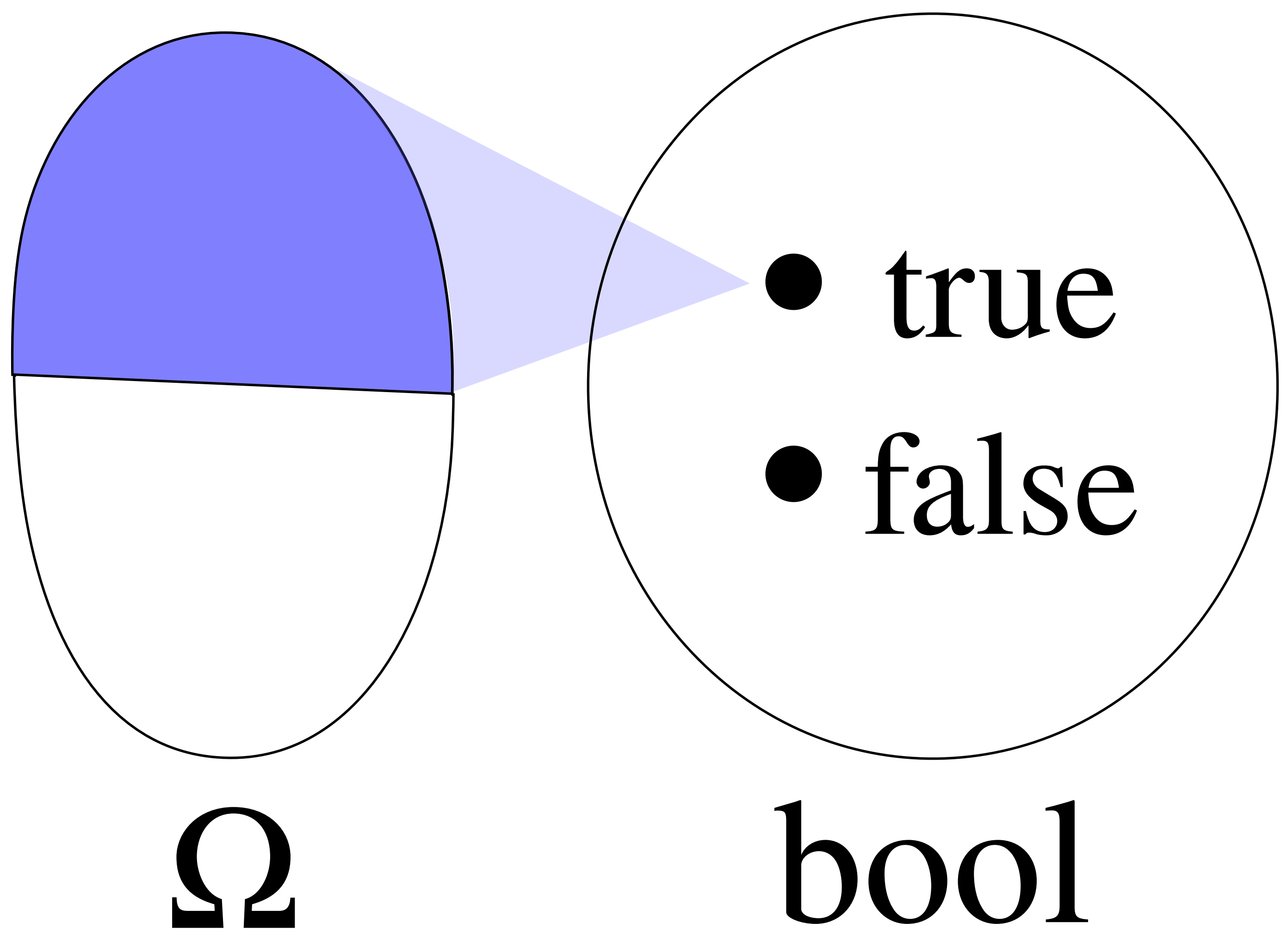

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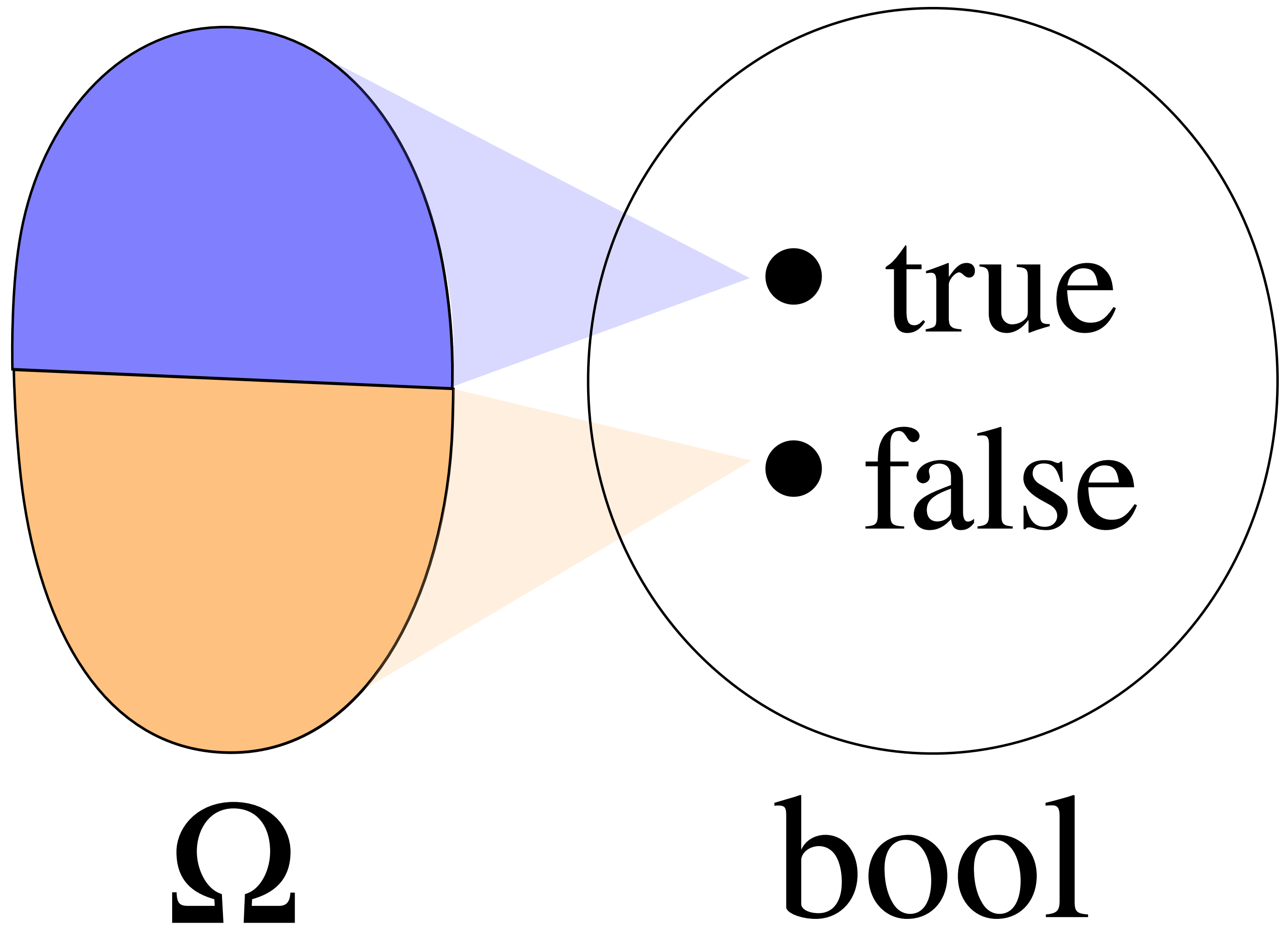
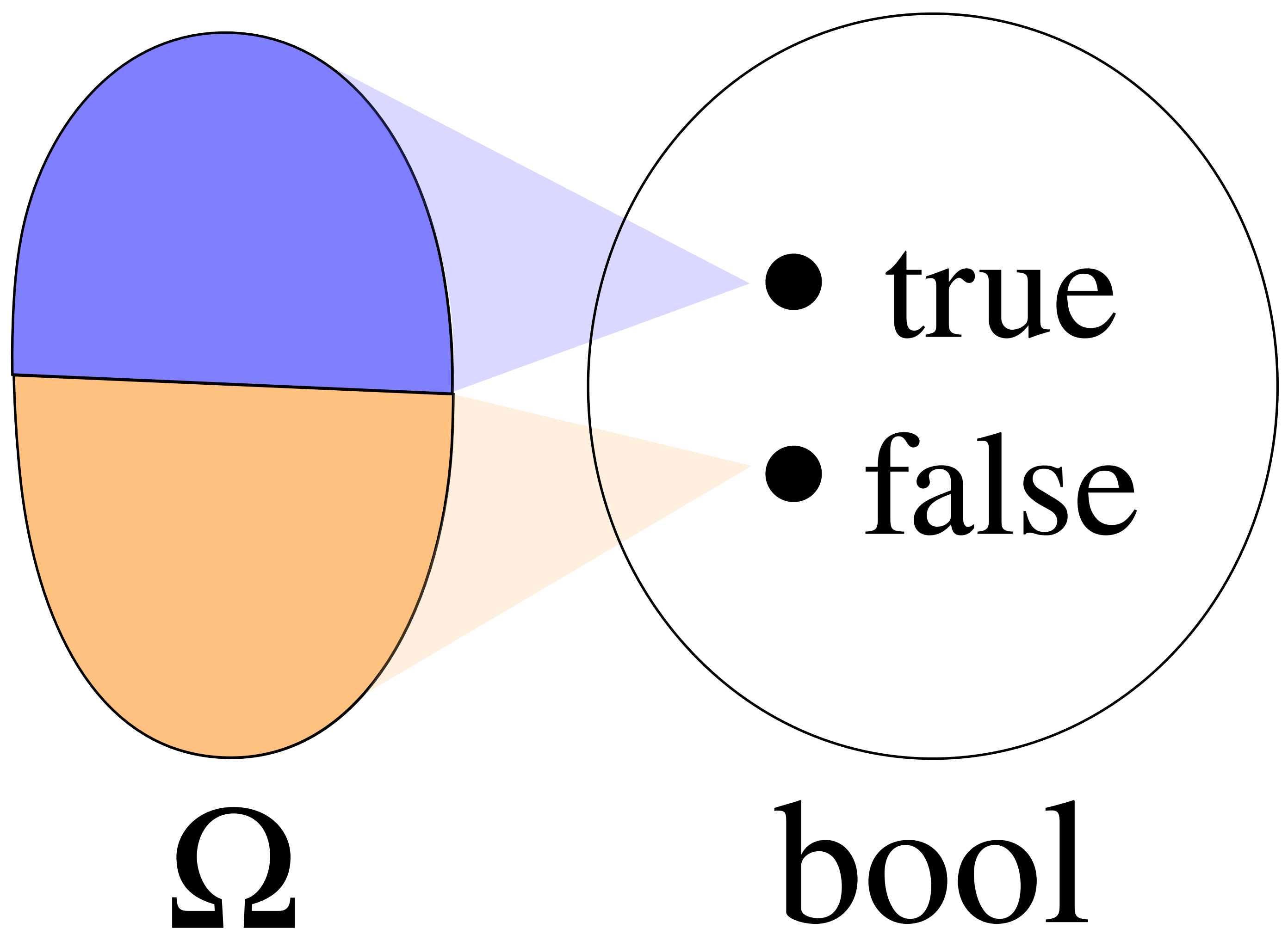


Probability spaces as heaps

$X \sim \mathrm{Ber}(1/2)$ really means...

$X : \Omega \to \mathrm{bool}$

μ : events $\to [0,1]$



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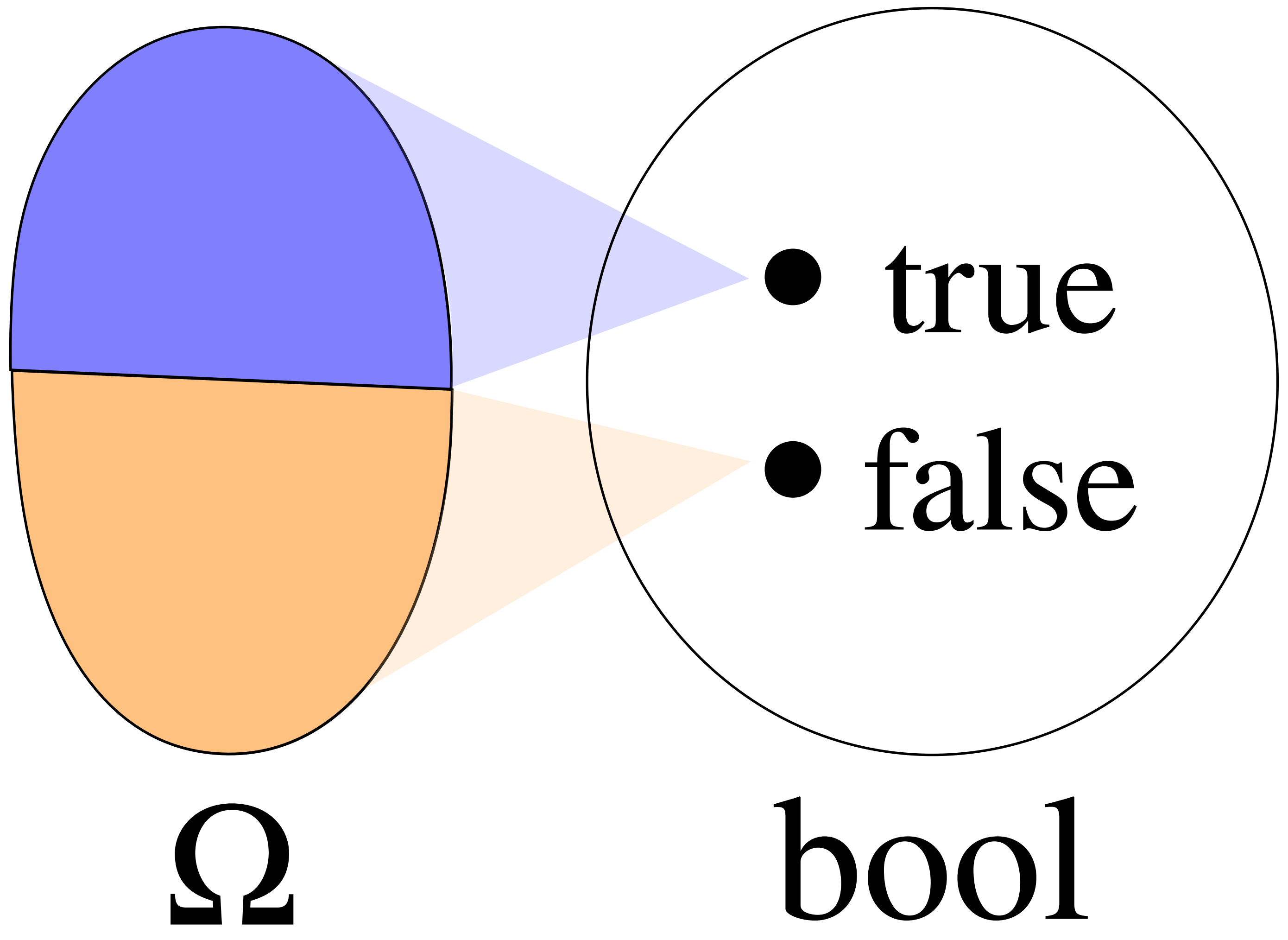
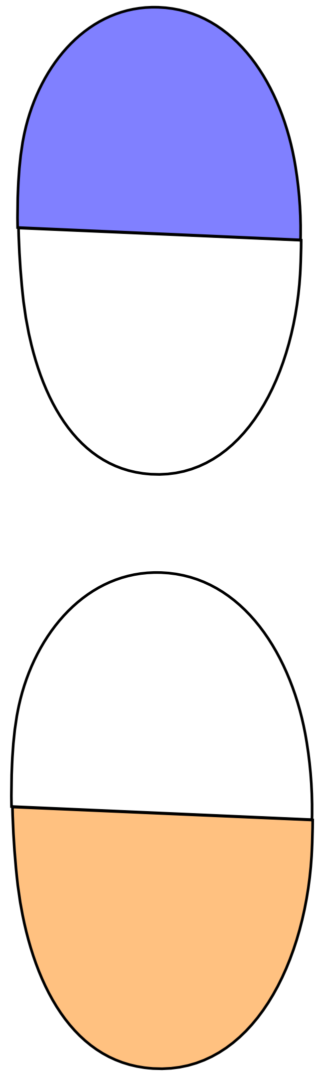


$\mu : \mathrm{events} \to [0,1]$



$\in$ events

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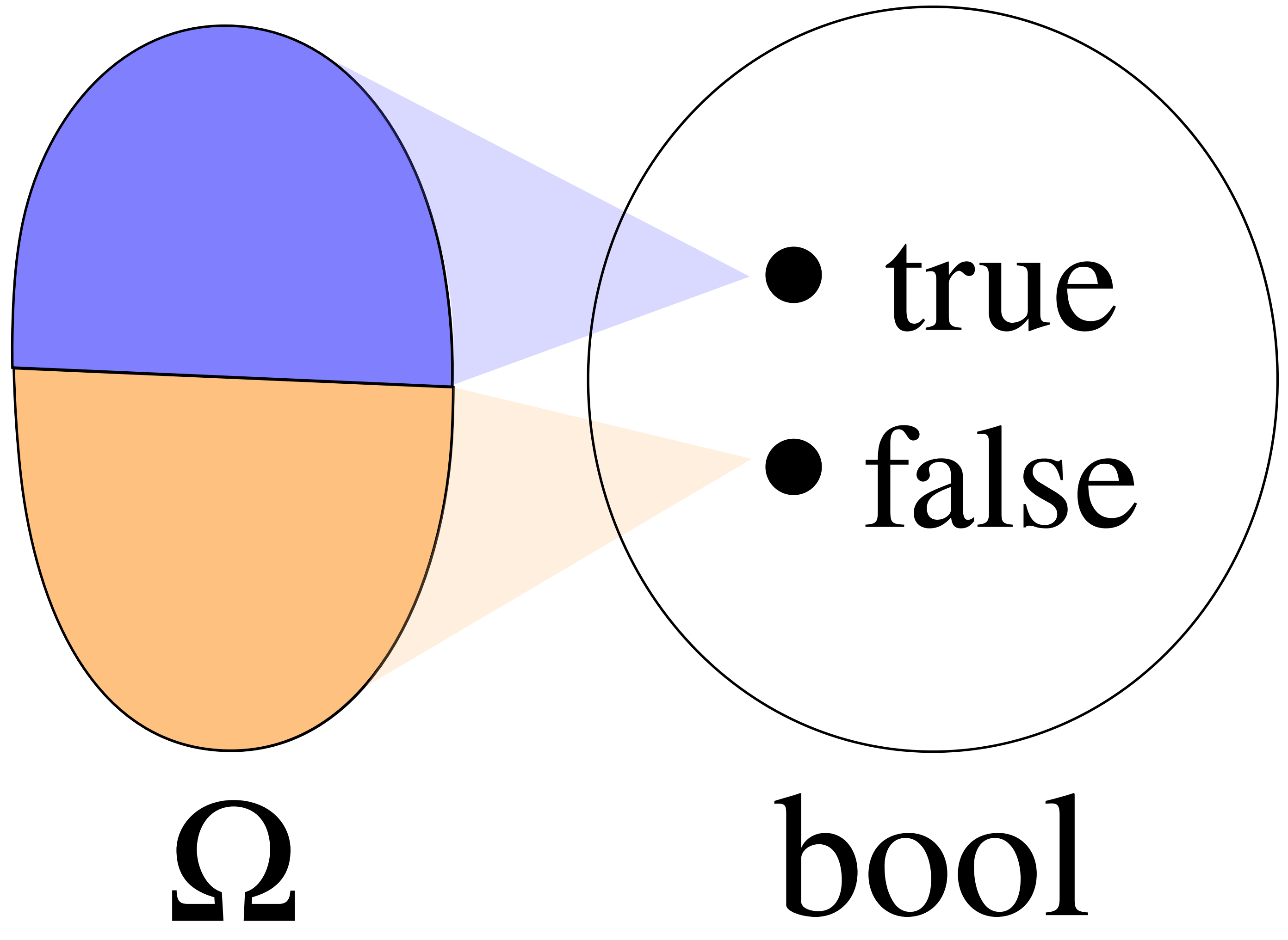

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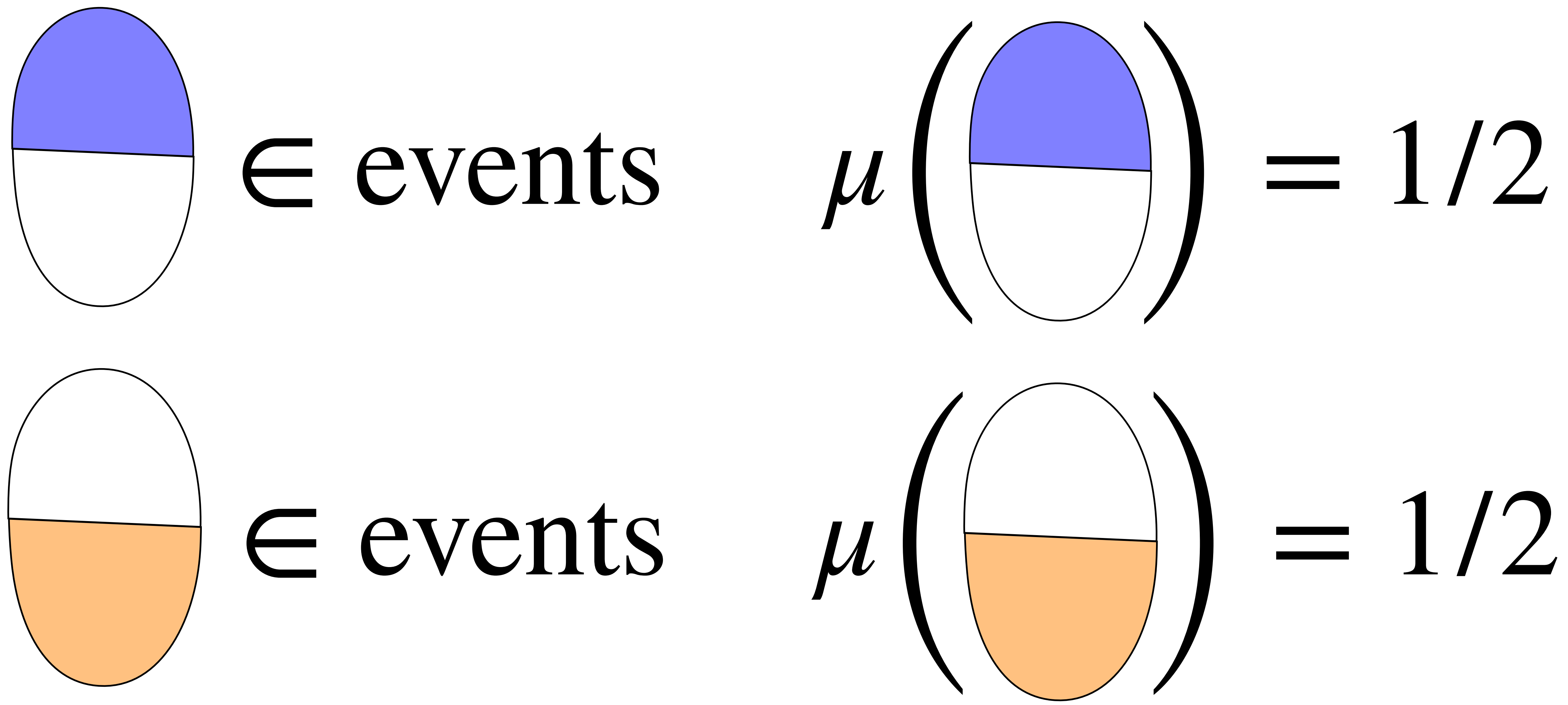

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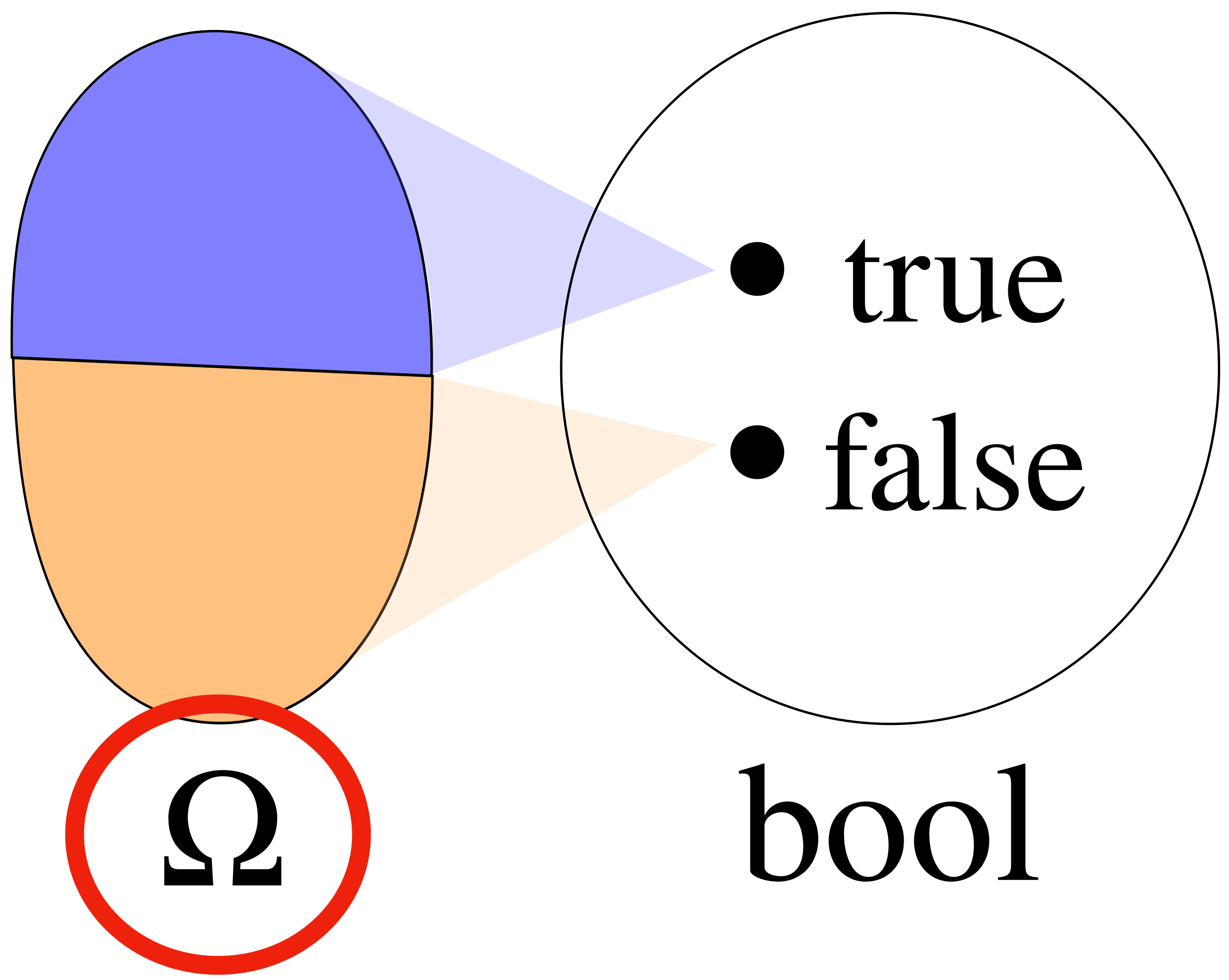

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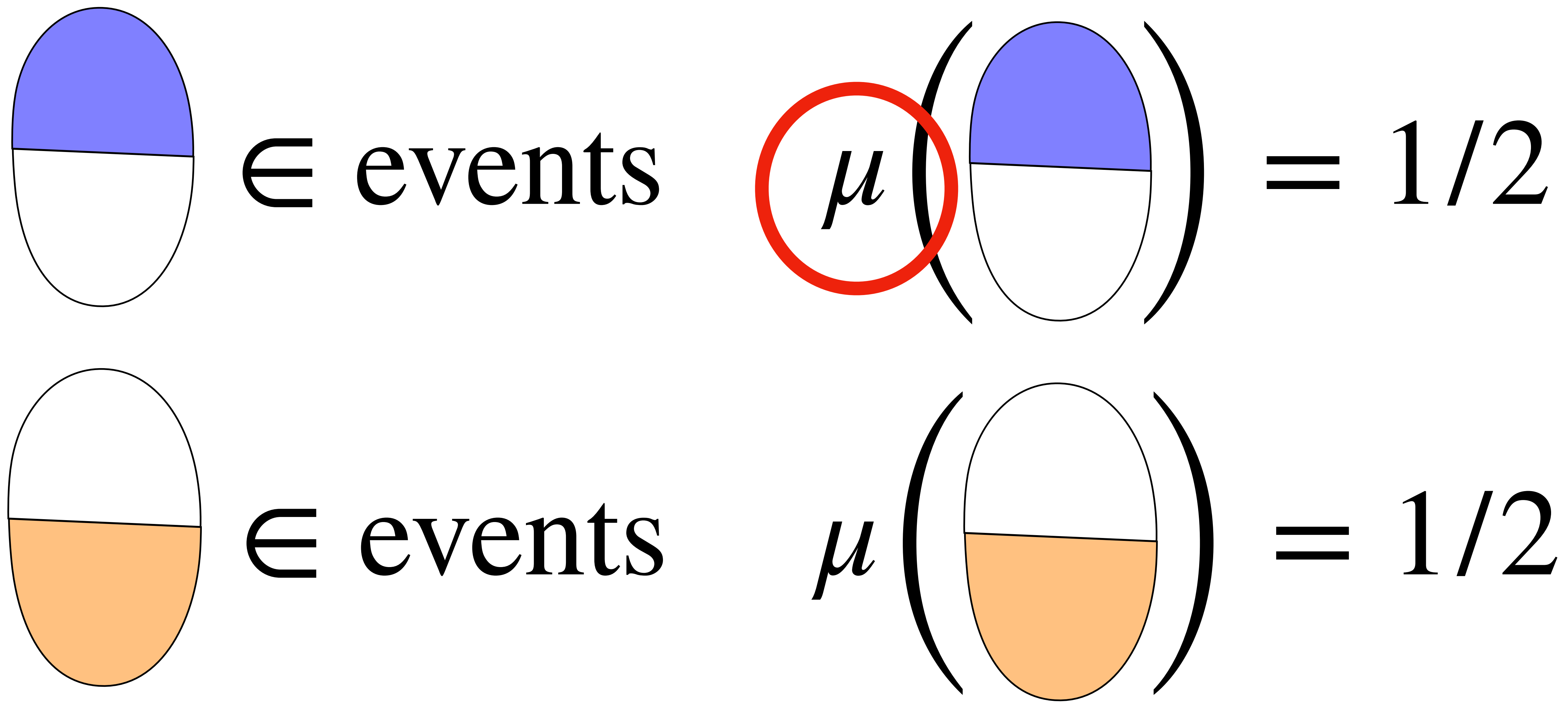

Probability spaces as heaps

$X \sim \mathrm{Ber}(1/2)$ really means...

events

μ

Ω

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Only accessed indirectly through X

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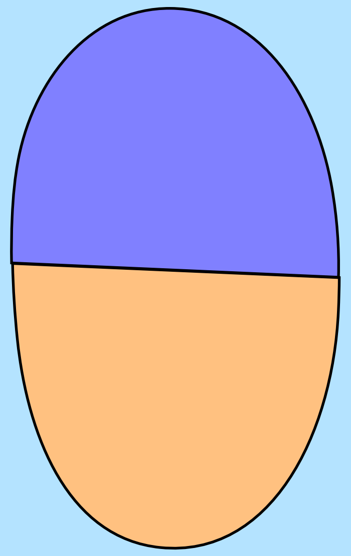
Together, form a probability space

Ω

Probability spaces as heaps

Probability theory

X

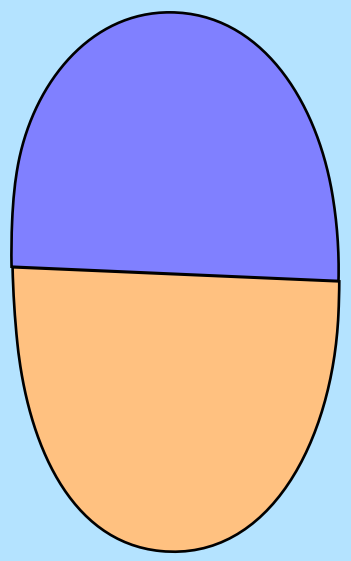


$(\Omega, \text{events}, \mu)$

Probability spaces as heaps

Probability theory $\simeq$ Mutable references

X



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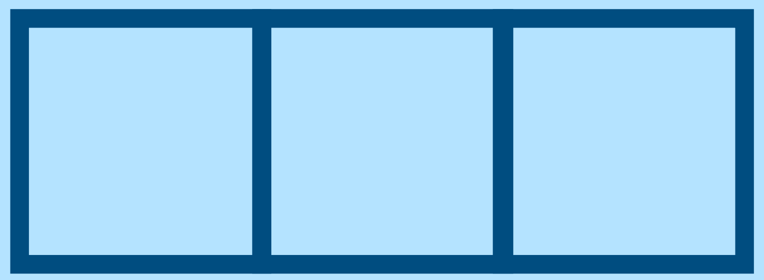
ℓ



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Key idea

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ℓ_x

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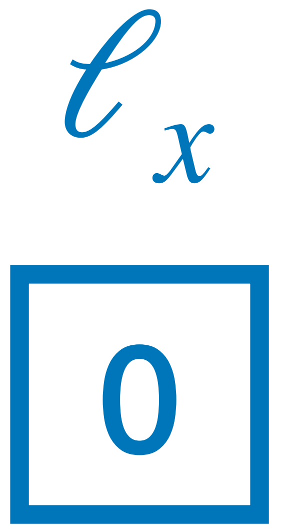

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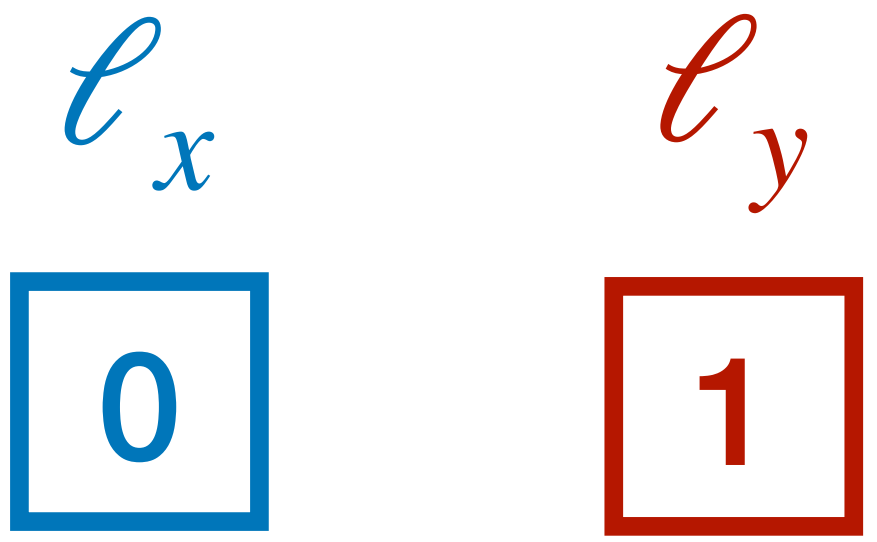

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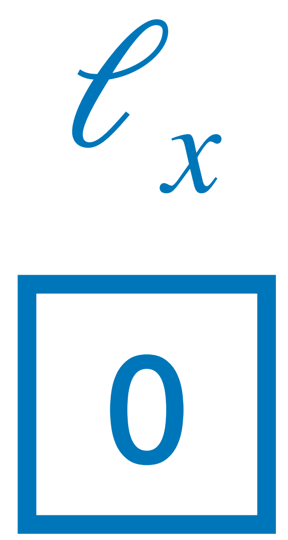

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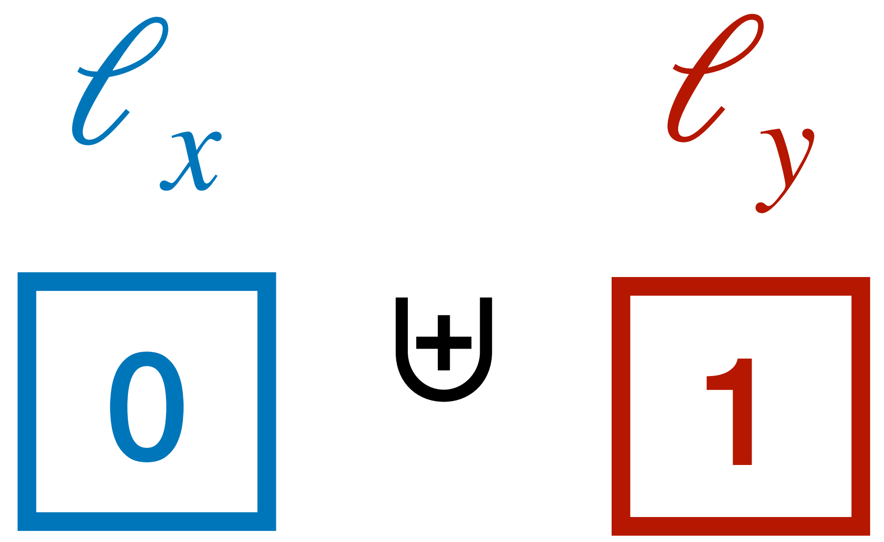

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- Probability spaces are the heaps of probability theory.

$$X \leftarrow \texttt{flip}\ 1/2;$$

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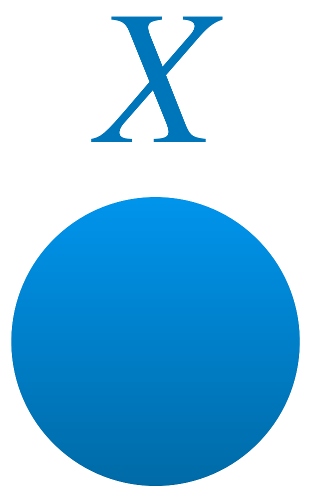

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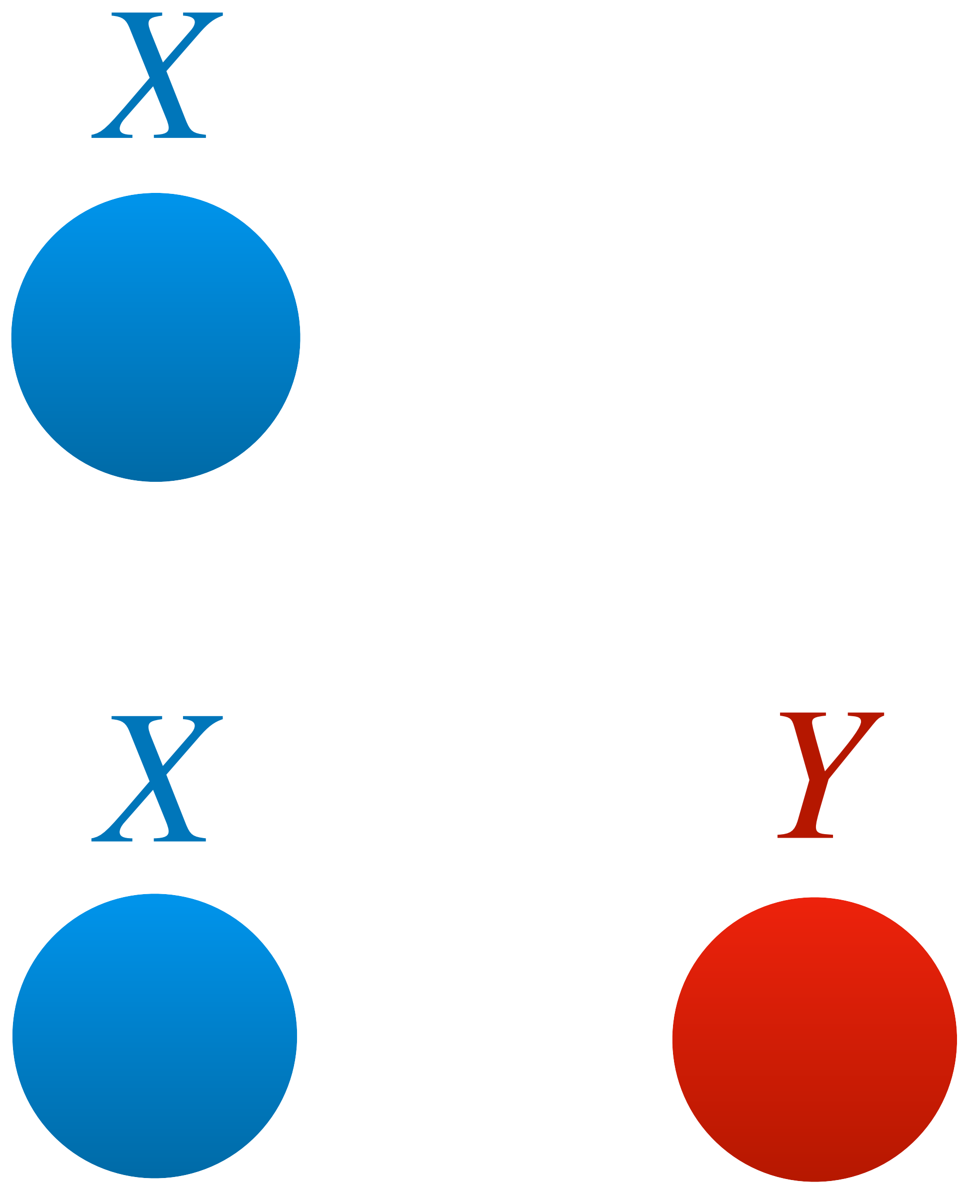

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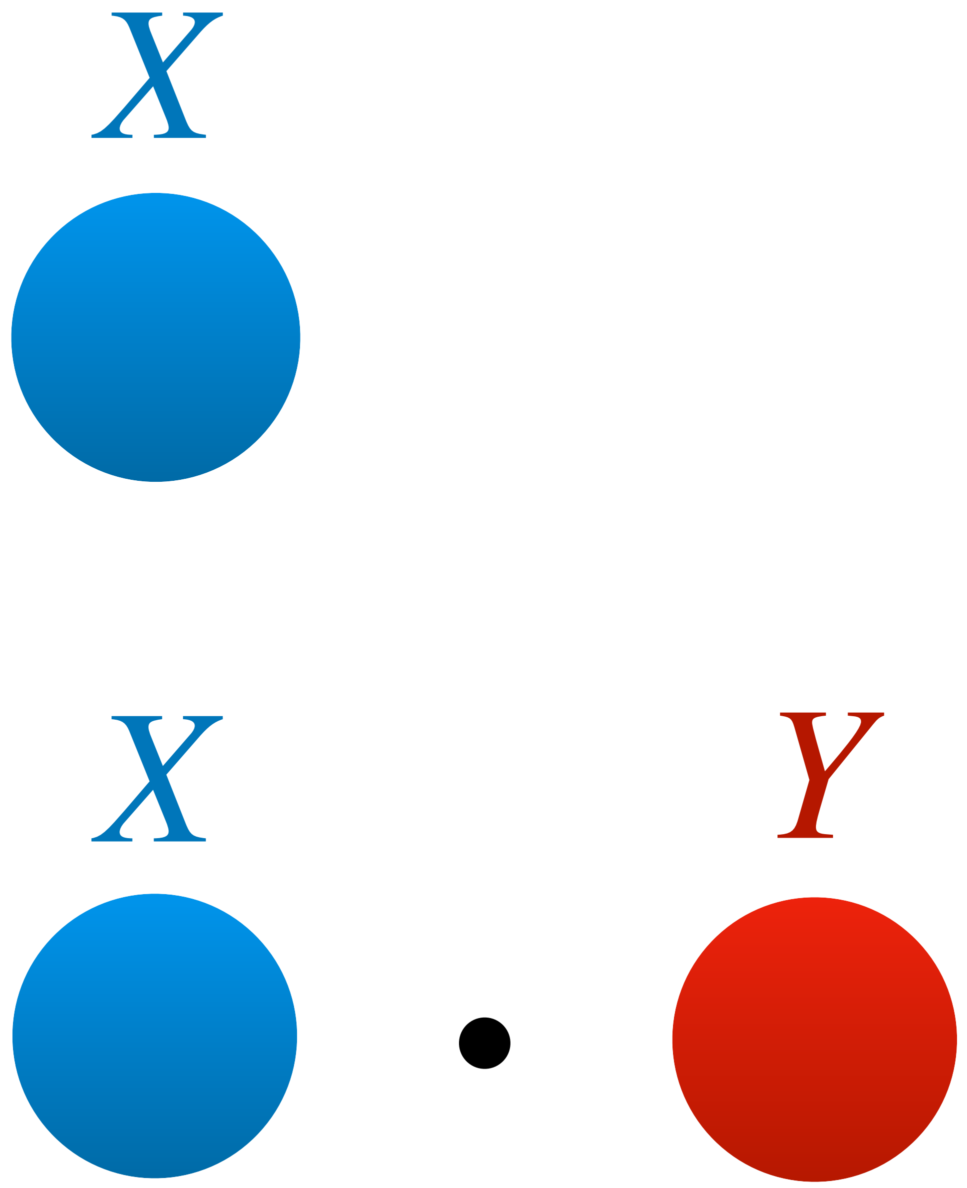

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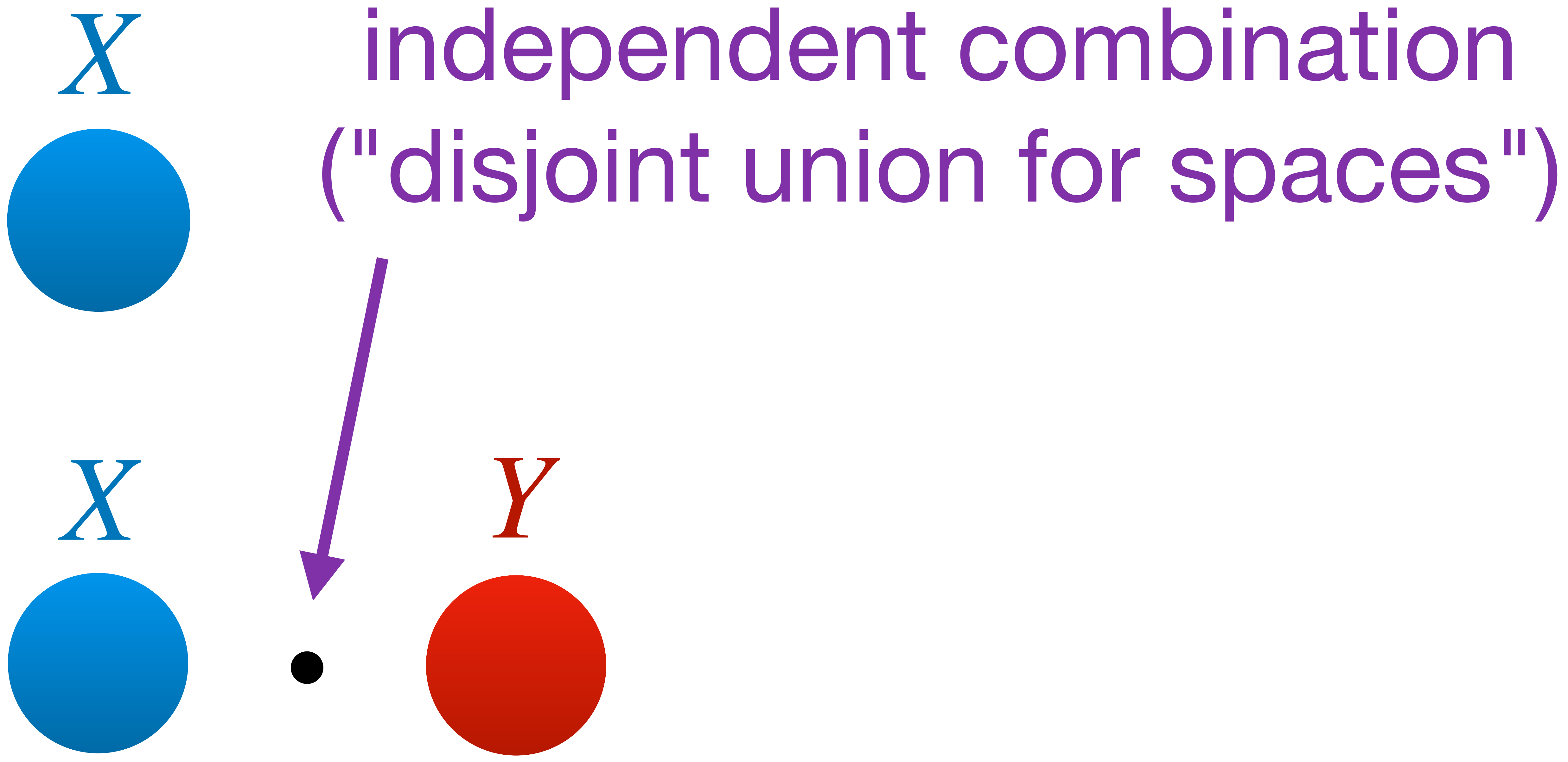

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- Separating conjunction decomposes probability spaces:

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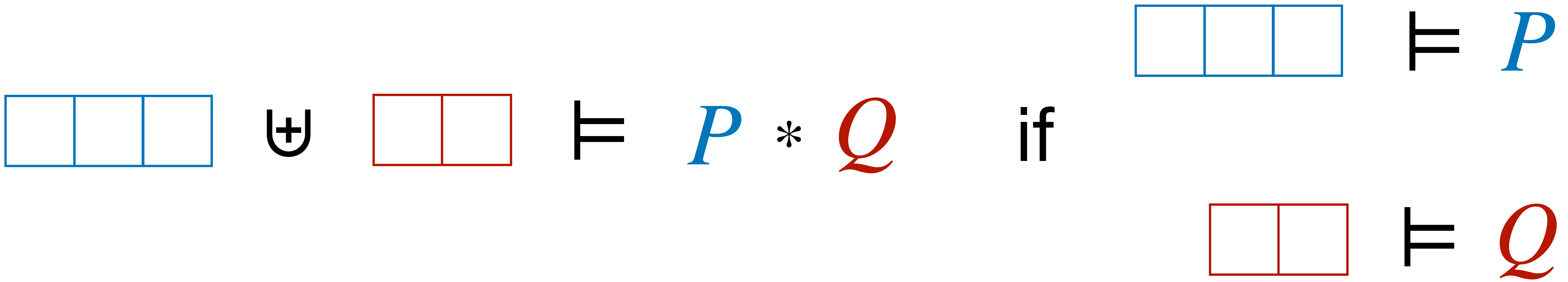

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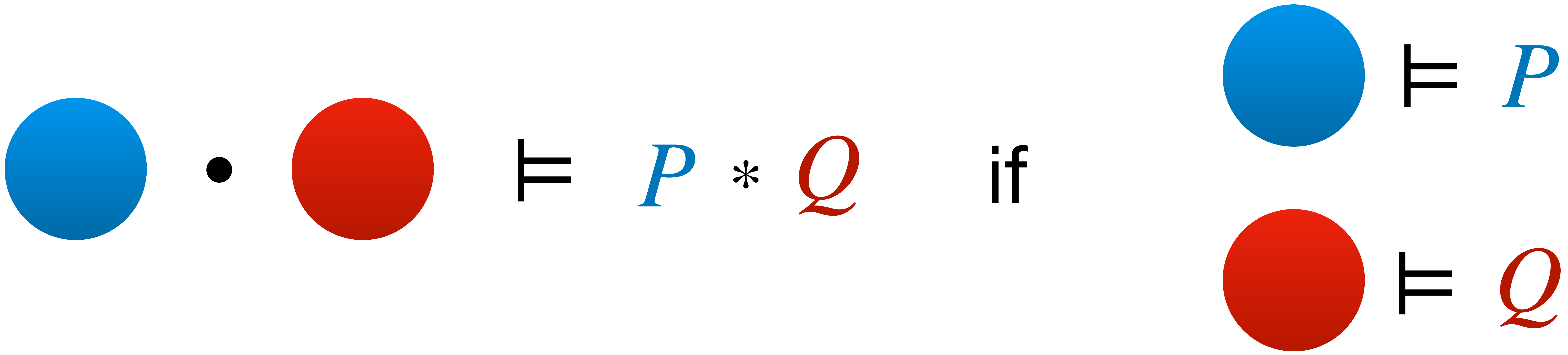

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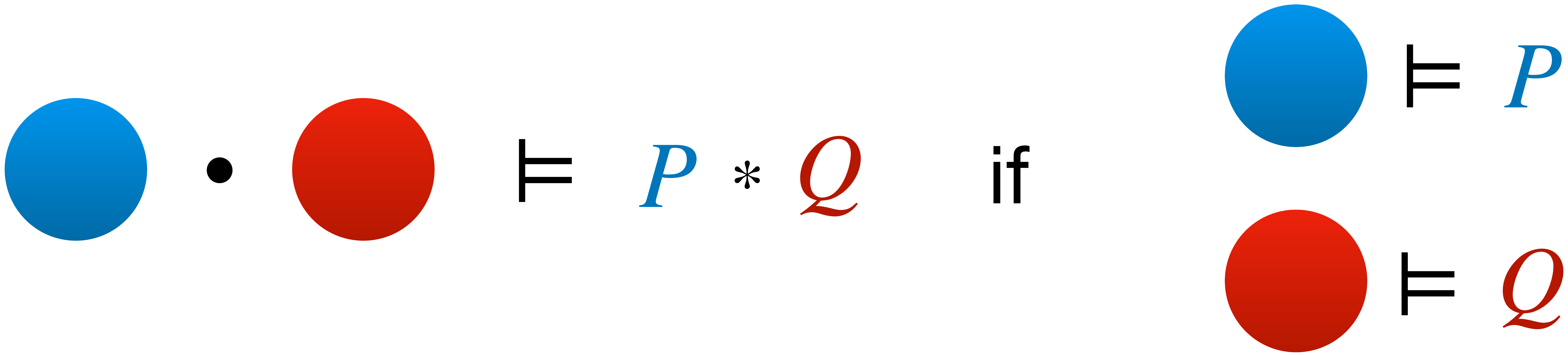

- $\Longrightarrow$ frame rule, star as independence, good interop, ...

Lilac: a modal separation logic for conditional probability

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- Conditioning as a *modality*:

Lilac: a modal separation logic for conditional probability

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$$\underset{x \leftarrow X}{\mathbf{C}} \; P$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$\underset{x \leftarrow X}{\mathbf{C}}\ P$$

P holds conditional on $X = x$ for all x

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

X and Y are independent

↓

$$X \sim \text{Ber}(1/2) \quad * \quad Y \sim \text{Ber}(1/2)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

X and Y are independent

$$X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2)$$

X and Y have distribution $\mathrm{Ber}(1/2)$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$\underset{z \leftarrow Z}{\mathbf{C}} \left(X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2) \right)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

X and Y are conditionally independent given Z

↓

$$\mathbf{C}_{z \leftarrow Z} \left(X \sim \mathrm{Ber}(1/2) \;\; * \;\; Y \sim \mathrm{Ber}(1/2) \right)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

X and Y are conditionally independent given Z

$$\mathbf{C}_{z \leftarrow Z}\left(X \sim \mathrm{Ber}(1/2) \ast Y \sim \mathrm{Ber}(1/2) \right)$$

X and Y have conditional distribution $\mathrm{Ber}(1/2)$ given Z

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$\Pr[E] = 1/2$ E has probability $1/2$

$\mathbf{E}[X] = 0$ X has expectation 0

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$\underset{x \leftarrow X}{\mathbf{C}}\left(\Pr[E] = 1/2\right)$$

E has probability $1/2$ given $X = x$

$$\mathbf{E}[X] = 0$$

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Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$\underset{x \leftarrow X}{\mathbf{C}}\left(\Pr[E] = 1/2\right)$$

E has probability $1/2$ given $X = x$

$$\underset{y \leftarrow Y}{\mathbf{C}}\left(\mathbf{E}[X] = 0\right)$$

X has conditional expectation 0

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*
- Laws express intuitive facts and standard theorems:

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*
- Laws express intuitive facts and standard theorems:

C-Total-Expectation

$$\underset{x \leftarrow X}{\mathbf{C}} \left(\mathbb{E}[E] = e \right) \;\wedge\; \mathbb{E}[e[X/x]] = v \vdash \mathbb{E}[E] = v$$

We used Lilac to verify

- Examples from prior work (cryptographic protocols)
- A tricky weighted sampling algorithm exercising
  - Continuous random variables
  - Quantitative reasoning
  - Separation as independence
  - Conditioning modality

Also in the paper

- Conditioning modality
- Ownership is measurability
- Worked examples
- Almost-sure equality $X =_{\text{a.s.}} Y$

Thanks!

Probability theory $\simeq$ Mutable references

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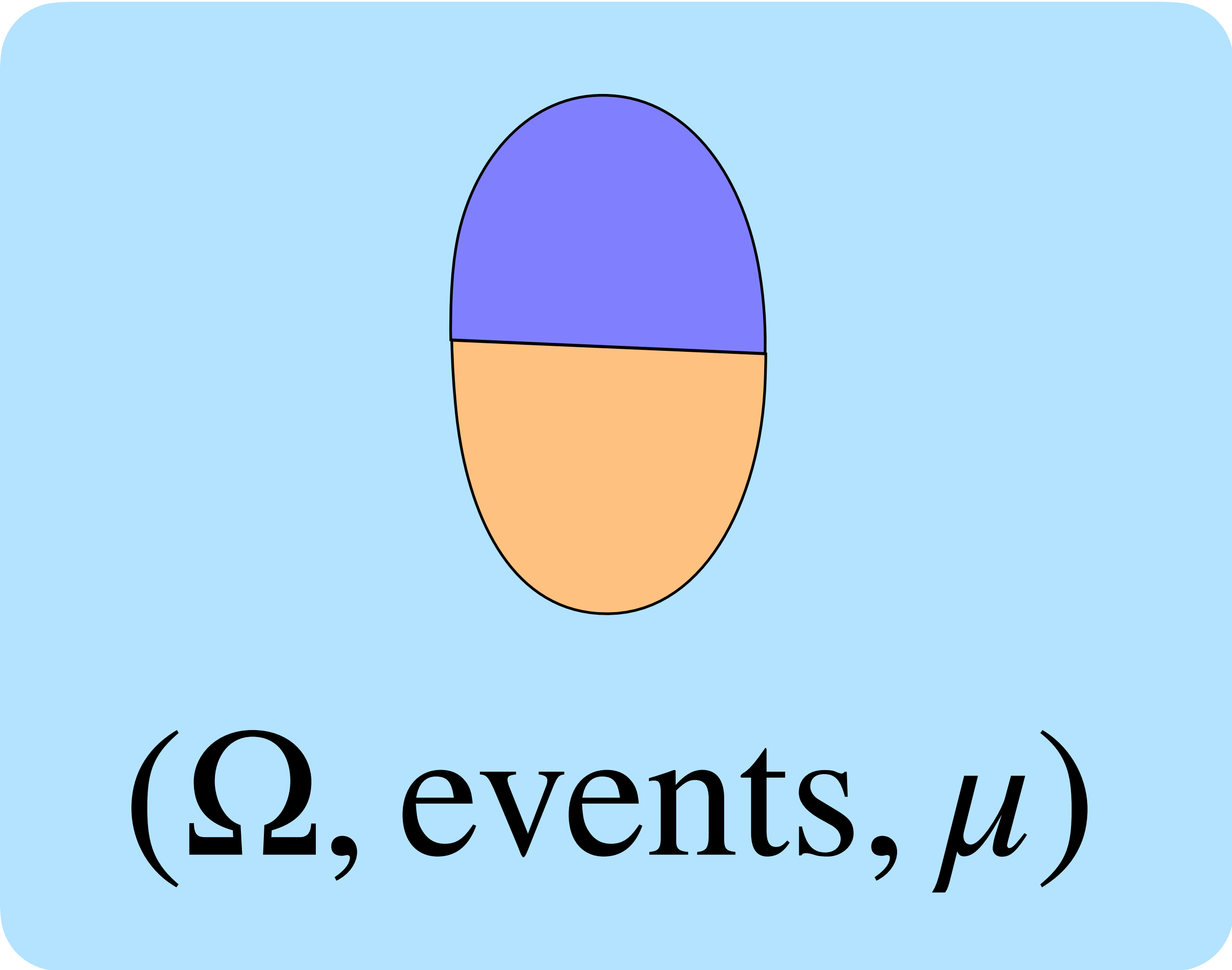

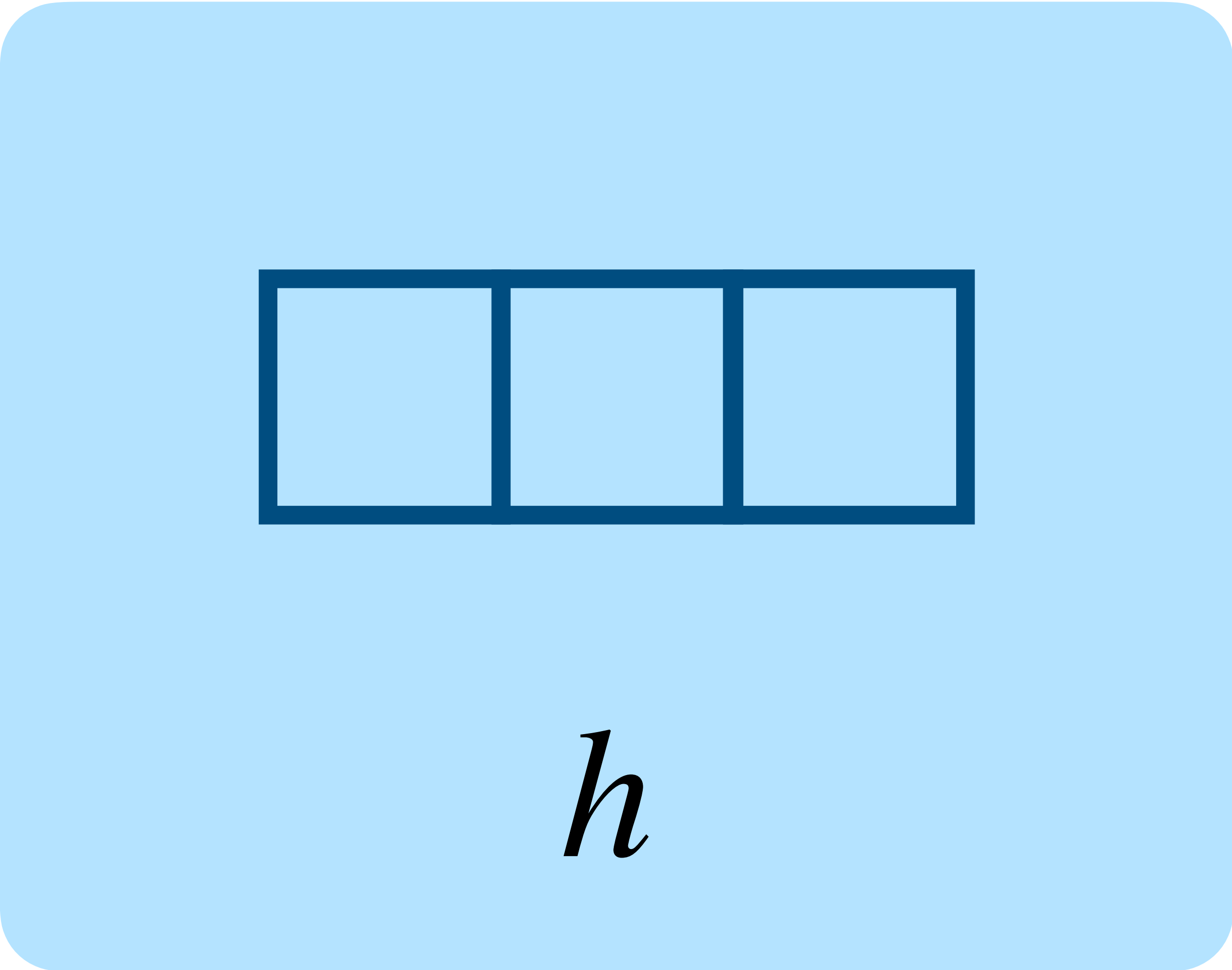

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